Additional Proofs for Joint User Grouping and Linear Virtual Beamforming: Complexity, Algorithms and Approximation Bounds

Mingyi Hong, Zi Xu, Meisam Razaviyayn and Zhi-Quan Luo

Abstract

In this document, we provide missing proofs for some of the results in the paper “Joint User Grouping and Linear Virtual Beamforming: Complexity, Algorithms and Approximation Bounds” [1].

I. PROOF OF CLAIM 2

Proof: Suppose on the contrary, there exists \( i \in \mathcal{M} \) such that \( \text{Tr}[D_i X^*] < 1 \). Then from the definition of \( D_i \) in (6) of [1], we have: \( \text{Tr}[C_{i,1} X_1^*] + \text{Tr}[C_{i,0} X_0^*] < 1 \), which is equivalent to: \( \frac{1}{p} X_1^*[i,i] < \frac{1}{2} + \frac{1}{p} X_0^*[i,M+1] \). Due to the assumption that the inequality is strict, we can find a constant \( \delta > 0 \) such that

\[
\frac{1}{p} X_1^*[i,i] + \delta = \frac{1}{2} + \frac{1}{2} X_0^*[i,M+1].
\]

As a result, let \( \hat{X}_1 = X_1^* + \delta e_i e_i^T \succeq 0 \), then \( \hat{X} = \text{blkdg}[X_0^*, \hat{X}_1] \) is also feasible for problem (SDP1). This alternative solution achieves the following objective

\[
\nu_{SDP}^*(\hat{X}) = \text{Tr}[R(X_1^* + \delta e_i e_i^T)] = \text{Tr}[RX_1^*] + \delta \text{Tr}[Re_i e_i^T] \geq \text{Tr}[RX_1^*],
\]

where in (a) we have used that fact that \( \delta > 0 \) and \( R[i,i] > 0 \) (due to the strict positive definiteness of \( R \)). Clearly, the inequality (1) is a contradiction to the optimality of \( X^* \).

In conclusion, we have

\[
\frac{1}{p} X_1^*[i,i] = \frac{1}{2} + \frac{1}{2} X_0^*[i,M+1], \quad \forall \ i = 1, \cdots, M.
\]

II. PROOF OF APPROXIMATION RATIO FOR SCHEDULING PROBLEM

Proof: Our goal is to show that there exits a \( \delta > 0 \) such that

\[
\text{Prob}\left(\min_{k=1,2} \{(w_k^{(t)})^H R w_k^{(t)}\} \geq \frac{1}{\alpha_2} \min_{k=1,2} \{\text{Tr}[R X_k^*]\}\right) \geq \delta > 0.
\]

M. Hong, M. Razaviyayn and Z.-Q. Luo are with the Department of Electrical and Computer Engineering University of Minnesota, Minneapolis, USA. Z. Xu is with the Department of Mathematics, Shanghai University, Shanghai, China.
Note that we have
\[ \text{Prob} \left( \min_{k=1,2} \{ (w_k^{(t)})^H R w_k^{(t)} \} \geq \frac{1}{\alpha_2} \min_{k=1,2} \{ \text{Tr}[R X_k^*] \} \right) \]
\[ = \text{Prob} \left( \min_{k=1,2} \left\{ \frac{1}{(t_k^{(t)})^2} \text{Tr}[R [S_k] Y_k^*] \right\} \geq \frac{1}{\alpha_2} \min_{k=1,2} \{ \text{Tr}[R X_k^*] \} \right) \]
\[ \geq \text{Prob} \left( \min_{k=1,2} \{ \text{Tr}[R [S_k] Y_k^*] \} \geq \frac{\beta}{\alpha_2} \min_{k=1,2} \{ \text{Tr}[R X_k^*] \}, \frac{1}{(t_1^{(t)})^2} \geq \frac{1}{\beta}, \frac{1}{(t_2^{(t)})^2} \geq \frac{1}{\beta} \right) . \]

We first lower bound \( \text{Tr}[R [S_2] Y_2^*] \) by \( \text{Tr}[R [S_2] Y_2^*] \geq \text{Tr}[R [S_2] X_2^*[S_2]] \), where the inequality is from the optimality of \( Y_2^* \) as well as the feasibility of \( X_2^*[S_2] \) to the problem (17) in [1]. The right hand side of the above inequality can be further lower bounded by
\[ \text{Tr}[R [S_2] X_2^*[S_2]] \geq \sum_{i \in S_2} X_2^*[i, i] \lambda_M(R) \]
\[ \overset{(i)}{=} \frac{P \lambda_M(R)}{2} \left( M - Q - \sum_{i \in S_2} X_0^*[i, M + 1] \right) \]
\[ \overset{(ii)}{\geq} \frac{P \lambda_M(R)}{2} \left( M - Q - \frac{(M - Q)Q}{M} \right) = \frac{P(M - Q)^2}{M} \lambda_M(R). \]
where (i) is because of the tightness of the constraint (18b) in [1]. We argue the inequality (ii) as follows. Due to Step S2) of the algorithm, \( \{ X_0^*[i, M + 1]\}_{i \in S} \) are the smallest \( M - Q \) elements in \( \{ X_0^*[i, M + 1]\}_{i = 1}^{M - 1} \). Combining with the fact that \( \sum_{i = 1}^{M - 1} X_0^*[i, M + 1] = Q \), we must have that \( \sum_{i \in S} X_0^*[i, M + 1] \leq \frac{M - Q}{M} Q \), which further implies (ii). Combining the above two inequalities, we obtain \( \text{Tr}[R [S_2] Y_2^*] \geq \frac{P(M - Q)^2}{M} \lambda_M(R) \).

Using the same argument, we can lower bound \( \text{Tr}[R [S_1] Y_1^*] \) by: \( \text{Tr}[R [S_1] Y_1^*] \geq \frac{P(M - Q)^2}{M} \lambda_M(R) \).

Combining the above two estimates, we have that
\[ \min_{k=1,2} \{ \text{Tr}[R [S_k] Y_k^*] \} \geq \min \{ Q^2, (M - Q)^2 \} \frac{P}{M} \lambda_M(R) . \]

Note that we further have
\[ \text{Tr}[R X_1^*] \leq \lambda_1(R) \text{Tr}[X_1^*] = \lambda_1(R) Q P \]
\[ \text{Tr}[R X_2^*] \leq \lambda_1(R) \text{Tr}[X_2^*] = \lambda_1(R) P(M - Q) \]
which implies \( \min_{k=1,2} \{ \text{Tr}[R X_k^*] \} \leq \lambda_1(R) P \min \{ Q, M - Q \} . \) Consequently, when choosing
\[ \beta = \frac{\lambda_1(R) P \min \{ Q, M - Q \}}{\min \{ Q^2, (M - Q)^2 \} \frac{P}{M} \lambda_M(R)} = \frac{\lambda_1(R)}{\lambda_M(R) \min \{ Q, M - Q \}} \]
we have
\[ \text{Prob} \left( \min_{k=1,2} \{ \text{Tr}[R [S_k] X_k^*] \} < \frac{\beta}{\alpha_2} \min_{k=1,2} \{ \text{Tr}[R X_k^*] \} \right) = 0 . \]
Going through similar steps as in Step 2)–Step 3) in the proof of Theorem 1, the final ratio is
\[
\alpha_2 = \frac{8M\lambda_1(R)}{\min\{Q, M - Q\} \lambda_M(R)} \ln(12 \max\{Q, M - Q\}).
\]
(4)
This completes the proof.

III. PROOF OF PROPOSITION 2

Proof: Below we show the case when \( R \) is diagonal. The claim is proved by using a polynomial time reduction from the equal partitioning with equal cardinality problem. Let \( M \) be a even number and set \( Q = \frac{M}{2} \). Given a vector \( c \in \mathbb{R}^M \) consists of positive elements \( c_1, \ldots, c_M \), let \( C = \sum_{i=1}^{M} c_i > 0 \), the equal partitioning with equal cardinality problem finds an index set \( I \) with \( |I| = \frac{M}{2} \) such that \( \frac{C}{2} = \sum_{i \in I} c_i \).

Let \( R = \text{diag}(c) \). We claim that determining if problem (CP2) can achieve an optimal value of \( \frac{C}{2} \) is NP-hard. Let \( I \) denote the set such that \( I = \{ i : a_i = 1 \} \) with \( |I| = \frac{M}{2} \). Then the objective of (CP2) can be written as
\[
\min \left\{ \sum_{i \in I} c_i |w_{1,i}|^2, \sum_{j \notin I} c_j |w_{2,j}|^2 \right\} \leq \min \left\{ P \sum_{i \in I} c_i, P \sum_{j \notin I} c_j \right\},
\]
where the inequality is achieved by setting \( |w_{1,i}|^2 = P, |w_{2,i}|^2 = 0, \forall i \in I, |w_{1,i}|^2 = 0, |w_{2,i}|^2 = P, \forall i \notin I \). Clearly, checking if we can find an index set \( I \) so that this problem can achieve an objective value of \( \frac{C}{2} \) is equivalent to finding a subset \( I \) with \( |I| = \frac{M}{2} \) satisfying \( \frac{C}{2} = \sum_{i \in I} c_i \), which is exactly the equal partitioning with equal cardinality problem.

The case when \( R \) is of rank 1 can be shown similarly.

IV. PROOF OF PROPOSITION 4

Proof: Assume that \( f_i = 1, \forall i \). It suffices to show that checking the feasibility problem
\[
\min_{k = 1, 2} \frac{w_k^H \mathbb{E}[gg^H] w_k}{\sigma_n^2 + w_k^H \text{diag} \left( \mathbb{E}[gg^H] \right) w_k} \geq t\]
\[
|w_{1,i}|^2 \left( P_0 \mathbb{E}[|f_i|^2] + \sigma_n^2 \right) \leq a_i P, |w_{2,i}|^2 \left( P_0 \mathbb{E}[|f_i|^2] + \sigma_n^2 \right) \leq (1 - a_i) P, i = 1, \ldots, M
\]
\[
\sum_{i = 1}^{M} a_i = Q, \quad a_i \in \{0, 1\}, i = 1, \ldots, M,
\]
is NP-hard. Let us also consider the following system parameters:
\[
\mathbb{E}[|g_i|^2] = c_i > 0, \quad P_0 = \sigma_n^2 = 1, \quad P = 2, \quad Q = \frac{M}{2}, \quad t = \frac{1}{2}, \quad \sigma_n^2 = \sum_{i} c_i / 2.
\]

Using the above parameters, one can easily show that problem (5) is feasible iff the following NP-complete equal partition with equal cardinality problem is feasible:

Find an index set \( I \subseteq \{1, 2, \ldots, M\} \) with \( |I| = \frac{M}{2} \) such that \( \sum_{i \in I} c_i = \sum_{i \in I^c} c_i \).

Such equivalence completes the proof.
REFERENCES